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Modeling and asset allocation for financial markets based on a discrete time microstructure model

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Abstract. On the basis of the market microstructure theory and the continuous time stochastic volatility-style microstructure model, a discrete time stochastic volatility microstructure model with state-observability is proposed for describing the dynamics of financial markets. From the discrete time microstructure model proposed, estimates of two immeasurable state variables representing the market excess demand and liquidity respectively may be obtained. A simple trading strategy for dynamic asset allocation, based on the indirectly obtained excess demand information instead of the prediction for price, is presented. An approach to the estimation of the discrete time microstructure model using the extended Kalman filter and the maximum likelihood method is also presented. Case studies on financial market modeling and the estimated model-based asset dynamic allocation control for the JPY/USD (Japanese Yen/US Dollar) exchange rate and Japan TOPIX (TOkyo stock Price IndeX) show satisfactory modeling precision and control performance.

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1 Introduction

It is well-known that financial markets generally display randomness, nonlinearity, jumps, and properties such as stochastic volatility. For this reason, Markov diffusion processes (mainly Wiener processes and/or Poisson processes) may be used as stochastic engines to construct mathematical models for characterizing financial markets. Many mathematical models have already been proposed to describe the dynamics of financial markets, including the nonlinear diffusion model, jump diffusion model, stochastic volatility model, and the general evolutionary model combining the features of all the above models (see *e.g.* [1]).

In discrete time modeling, the ARCH (autoregressive conditional heteroskedasticity) based stochastic volatility-style models developed by Engle [2], Bollerslev [3] and Nelson [4] were rapidly adopted in econometrics, financial economics and microeconomics, largely because the conditional likelihood function for ARCH-style models is easily calculated. However, most of the models proposed focus on modeling the dynamics of asset price itself and especially its conditional variance. In fact, it is also necessary to deal with the dynamics of a financial market from different points of view, as has been proposed by some researchers who have developed phenomenological models based on identifying different processes influencing the demand and supply of the market. One of the most interesting of these models is the microstructure model proposed by Bouchaud and Cont [5] on the basis of the market microstructure theory (see *e.g.* [6]). The model is defined by

$$\mathrm{d}P_t = \lambda \phi_t \, \mathrm{d}t \tag{1}$$

where P_t is the asset price, λ is the (inverse of) market liquidity, and ϕ_t is the excess demand which is defined by $\phi_t = \phi_t^+ - \phi_t^-$, where ϕ_t^+ is the instantaneous demand and ϕ_t^- is the instantaneous supply at any given instant of time for the asset. ϕ_t characterizes whether the market is over-valued ($\phi_t > 0$, which tends to push the price up) or under-valued ($\phi_t < 0$, which tends to push the price down). Model (1) assumes that price P_t is driven by the excess demand ϕ_t, and the amplitude of price changes is dependent on the liquidity of the market, *i.e.* $1/\lambda$. Therefore, if the liquidity is higher, then the market may absorb excess demand by means of small price changes, whereas when the liquidity is lower, a smaller excess demand may lead to larger price changes. However, model (1) only provides an abstract description for the dynamics of market, because the two market state variables (λ and ϕ_t) cannot easily be measured directly. If these variables could be estimated in real time from a time series of price, it would help

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traders make selling or buying decisions in market trading, and its impact on financial practice could be enormous.

Iino and Ozaki [7] proposed a set of continuous time stochastic differential equations following the approach of model (1). Their model was called the continuous time microstructure model and is given by

$$\begin{cases} \mathrm{d}P_t = \lambda_t\phi_t\,\mathrm{d}t + \lambda_t\,\mathrm{d}W_{1,t} \\ \mathrm{d}\phi_t = (\alpha_1 + \beta_1\phi_t)\mathrm{d}t + \gamma_1\,\mathrm{d}W_{2,t} \\ \mathrm{d}\log\lambda_t = (\alpha_2 + \beta_2\log\lambda_t)\mathrm{d}t + \gamma_2\,\mathrm{d}W_{3,t} \end{cases} \tag{2}$$

where $W_{1,t}$, $W_{2,t}$ and $W_{3,t}$ are independent Wiener processes, and α_1, β_1, γ_1, α_2, β_2, and γ_2 are constant parameters. Model (2) treats the hidden excess demand ϕ_t and market liquidity λ_t as immeasurable state variables, which may be estimated by filtering techniques, and expresses the variation of conditional variance of the price, the most prominent characteristic of financial markets, by the change of market liquidity. Considering the market mechanism, such a relation between the liquidity and the conditional variance is natural. Therefore, model (2) offers more useful information about the internal characteristics of a price-varying process than can be gained just by looking at the price data itself or its prediction.

Ozaki *et al.* [8] presented an asset allocation control method for foreign exchange based on the estimated excess demand $\hat{\phi}_{t|t}$ offered by model (2). The continuous time stochastic differential equations model (2) was estimated using the local linearization technique [7,9–11]. However, the computational load of estimating the continuous time model (2) is usually quite large, and small changes to the model may lead to complicated computations, thus making the method inconvenient for some applications.

In this paper, we propose a discrete time microstructure (DTMS) model describing the stochastic volatility dynamics of financial markets. A DTMS state space model is developed from the continuous time microstructure model (2). The conditional mean, conditional variance and state observability condition are considered simultaneously in deriving the estimates of the DTMS model parameters. The DTMS model is estimated by using the extended Kalman filter and the maximum likelihood method. Model estimation results are obtained from the proposed DTMS similar to those obtained from the continuous time microstructure model (2), but the computational burden is considerably reduced. Furthermore, the DTMS model and its estimation method are much more flexible, allowing the model to be easily modified for different applications. An improved indirect feedback control approach, using the DTMS model for asset dynamic allocation on the basis of estimated excess demand $\hat{\phi}_{t|t}$ is also presented. Section 5 shows two examples of modeling real data and dynamic asset allocation for the JPY/USD currency exchange rate and the Japan TOPIX index. It may be seen how real-time estimated excess demand may be useful for traders considering dynamic asset allocation processes. In both cases, satisfactory trading strategies can be implemented in terms of the obtained profit.

2 Discrete time microstructure model

Using Euler's discrete time approximation to model (2) and directly modeling the conditional variance λ_t^2 of the price P_t, the DTMS (discrete time microstructure) model is derived as follows

$$\begin{cases} P_k = P_{k-1} + \lambda_{k-1}\phi_{k-1} + \gamma_3\lambda_{k-1}\xi_{1,k} \\ \phi_k = \alpha_1 + (1+\beta_1)\phi_{k-1} + \gamma_1\xi_{2,k} \\ \log\lambda_k^2 = \alpha_2 + (1+\beta_2)\log\lambda_{k-1}^2 + \gamma_2\xi_{3,k} \end{cases} \tag{3}$$

where $\xi_{1,k}$ $N(0,1)$, $\xi_{2,k}$ $N(0,1)$ and $\xi_{3,k}$ $N(0,1)$ are independent white noise processes. Note that the constant γ_3 in the discretized model (3) is a new addition that helps to describe the relation between P_k, λ_{k-1} and ϕ_{k-1} more appropriately. Thus model (3) may be also regarded as a redesigned discrete time microstructure model. To estimate model (3) by the Kalman filtering technique and the maximum likelihood method, we need to build a state space equation model with state-observability. First, from model (3), the state equation of the DTMS model may be given as follows

$$\mathbf{X}_k = \mathbf{A}(\mathbf{X}_{k-1}|\boldsymbol{\theta})\mathbf{X}_{k-1} + \boldsymbol{\Omega}_k,\ \boldsymbol{\Omega}_k \sim N(0, \mathbf{Q}_k) \tag{4}$$

where

$$\mathbf{X}_k = \begin{bmatrix} P_k & \phi_k & \log\lambda_k^2 \end{bmatrix}^T$$

$$\mathbf{A}(\mathbf{X}_{k-1}|\boldsymbol{\theta}) = \begin{bmatrix} 1 & \lambda_{k-1} & 0 \\ \dfrac{\alpha_1}{P_{k-1}} & 1+\beta_1 & 0 \\ \dfrac{\alpha_2}{P_{k-1}} & 0 & 1+\beta_2 \end{bmatrix}$$

$$\mathbf{Q}_k = \begin{bmatrix} \gamma_3^2\lambda_{k-1}^2 & 0 & 0 \\ 0 & \gamma_1^2 & 0 \\ 0 & 0 & \gamma_2^2 \end{bmatrix}$$

and $\boldsymbol{\theta}$ in (4) includes all the constant parameters to be estimated in model (3). In state equation (4), only price P_k is measurable. It is easy to check that the state space equation, composed of state equation (4), and the output equation including only P_k as observation variable (so that only the conditional mean is considered) is state-unobservable. We must therefore extract more information on the observations by considering the conditional mean and the conditional variance simultaneously so that the state-observability condition is satisfied.

Squaring both sides of the first equation in model (2), and ignoring the higher order terms of $\mathrm{d}t$ in accordance with the rules of Ito calculus [12], and noting that $(\mathrm{d}W_{1,t})^2$ behaves like $\mathrm{d}t$ in continuous time gives

$$(\mathrm{d}P_t)^2 = \lambda_t^2\,\mathrm{d}t. \tag{5}$$

Discretizing the above formula (5) and taking logarithms, we obtain

$$\log(P_{k+1}-P_k)^2 \approx \log\lambda_k^2. \tag{6}$$

If P_k and $\log(P_k-P_{k-1})^2$ are regarded as two observation variables and relation (6) is applied, a state space representation of DTMS model (3) satisfying state observability condition could be built as follows

$$\begin{cases}\mathbf{X}_{k+1}=\mathbf{A}(\mathbf{X}_k|\boldsymbol{\theta})\mathbf{X}_k+\boldsymbol{\Omega}_{k+1},\ \boldsymbol{\Omega}_{k+1}\sim N(0,\mathbf{Q}_{k+1})\\ \mathbf{Y}_k=\mathbf{C}(\mathbf{X}_k|\boldsymbol{\theta})\mathbf{X}_k+\boldsymbol{\Gamma}_k, \qquad \boldsymbol{\Gamma}_k\sim N(0,\mathbf{R}_k)\end{cases} \tag{7}$$

where

$$\mathbf{X}_k=\left[\,P_k\ \phi_k\ \log\lambda_k^2\,\right]^T$$

$$\mathbf{Y}_k=\left[\,P_k\ \log(P_k-P_{k-1})^2\,\right]^T$$

$$\mathbf{A}(\mathbf{X}_k|\boldsymbol{\theta})=\begin{bmatrix}1 & \lambda_k & 0\\ \dfrac{\alpha_1}{P_k} & 1+\beta_1 & 0\\ \dfrac{\alpha_2}{P_k} & 0 & 1+\beta_2\end{bmatrix}$$

$$\mathbf{C}(\mathbf{X}_k|\boldsymbol{\theta})=\begin{bmatrix}1 & 0 & 0\\ \dfrac{\delta}{P_k} & 0 & 1\end{bmatrix}$$

$$\mathbf{Q}_{k+1}=\begin{bmatrix}\gamma_3^2\lambda_k^2 & 0 & 0\\ 0 & \gamma_1^2 & 0\\ 0 & 0 & \gamma_2^2\end{bmatrix}$$

$$\mathbf{R}_k=\begin{bmatrix}\varepsilon_1^2 & 0\\ 0 & \varepsilon_2^2\end{bmatrix}$$

where ε_1, ε_2 are constants, and δ is a constant which adjusts the bias from the variable transformation in (6). We can see that there are very clear and direct relationships between the coefficient matrix elements of the state space equation (7) and the parameters of DTMS model (3). This contrasts with the continuous time model (2), where after local linearization, the matrix elements of the derived discrete time state space equation have a very complicated structure, which cannot even be represented by an explicit analytic formulas. The discrete time microstructure model (7) is thus computationally more efficient and easier to generalize to more complicated models.

3 Estimation method

The purpose here is first to determine all the constant parameters in DTMS model (7), and then to estimate the state $\mathbf{X}_{k|k-1}=E(\mathbf{X}_k|\mathbf{Y}_{k-1},...,\mathbf{Y}_1)$ from the observations (outputs) using model (7). This is a type of nonlinear filtering problem, for which we use the maximum likelihood method and the extended Kalman filter to estimate the parameters $\boldsymbol{\theta}$ and the state $\mathbf{X}_{k|k-1}$.

3.1 The likelihood function for the extended nonlinear Kalman filter

From the output equation of model (7), the estimated innovation of $\{\mathbf{Y}_k; k=1,2,...,N\}$ (where N is the number of data points) based on $\mathbf{X}_{k|k-1}$ is given by

$$\boldsymbol{\Gamma}_k=\mathbf{Y}_k-\mathbf{C}(\mathbf{X}_{k-1}|\boldsymbol{\theta})\mathbf{X}_{k|k-1}. \tag{8}$$

Assuming $\boldsymbol{\Gamma}_k$ is a 2-dimensional Gaussian white noise vector with covariance matrix $\boldsymbol{\Psi}_k$, then the joint conditional density of $\boldsymbol{\Gamma}_k$ may be written as

$$p(\boldsymbol{\Gamma}_k|\mathbf{Y}_{k-1},...,\mathbf{Y}_1,\boldsymbol{\theta})=\frac{1}{2\pi|\boldsymbol{\Psi}_k|^{1/2}}\exp\left(-\frac{1}{2}\boldsymbol{\Gamma}_k^T\boldsymbol{\Psi}_k^{-1}\boldsymbol{\Gamma}_k\right)$$

where $|\cdot|$ denotes the determinant [13]. Therefore (-2)log-likelihood of model (7) may be derived as follows

$$\begin{aligned}(-2)\log p(\mathbf{Y}_N,...,\mathbf{Y}_1|\boldsymbol{\theta})&=\sum_{k=1}^{N}(-2)\log\,p(\mathbf{Y}_k|\mathbf{Y}_{k-1},...,\mathbf{Y}_1,\boldsymbol{\theta})\\&=\sum_{k=1}^{N}(-2)\log\,p(\boldsymbol{\Gamma}_k|\mathbf{Y}_{k-1},...,\mathbf{Y}_1,\boldsymbol{\theta})\\&=\sum_{k=1}^{N}\left\{\log|\boldsymbol{\Psi}_k|+\boldsymbol{\Gamma}_k^T\boldsymbol{\Psi}_k^{-1}\boldsymbol{\Gamma}_k\right\}+2N\log 2\pi.\end{aligned} \tag{9}$$

3.2 Extended nonlinear Kalman filtering

If $\mathbf{A}(\hat{\mathbf{X}}_{k|k}|\boldsymbol{\theta})$ and $\mathbf{C}(\hat{\mathbf{X}}_{k|k}|\boldsymbol{\theta})$ are used to approximate $\mathbf{A}(\mathbf{X}_k|\boldsymbol{\theta})$ and $\mathbf{C}(\mathbf{X}_k|\boldsymbol{\theta})$ at each step respectively, then based on the conventional Kalman filtering approach (see *e.g.* [13] and [14]), the estimate $\hat{\mathbf{X}}_{k|k-1}$ may be calculated using the following extended recursive Kalman filtering scheme.

Prediction

Let $\hat{\mathbf{X}}_{k|k-1}$, $\mathbf{S}_k$ denote the conditional mean and conditional covariance of $\mathbf{X}_k$ given $\mathbf{Y}^{k-1}=\{\mathbf{Y}_1,...,\mathbf{Y}_{k-1}\}$, then the prediction and the innovation may be derived as follows

$$\begin{cases}\hat{\mathbf{X}}_{k|k-1}=E\left\{\mathbf{X}_k|\mathbf{Y}^{k-1}\right\}=\mathbf{A}\left(\hat{\mathbf{X}}_{k-1|k-1}\middle|\boldsymbol{\theta}\right)\hat{\mathbf{X}}_{k-1|k-1}\\ \hat{\boldsymbol{\Gamma}}_k=\mathbf{Y}_k-\mathbf{C}\left(\hat{\mathbf{X}}_{k-1|k-1}\middle|\boldsymbol{\theta}\right)\hat{\mathbf{X}}_{k|k-1}\\ \mathbf{S}_k=E\left\{\left(\mathbf{X}_k-\hat{\mathbf{X}}_{k|k-1}\right)\left(\mathbf{X}_k-\hat{\mathbf{X}}_{k|k-1}\right)^T\right\}\\ \quad=\mathbf{A}\left(\hat{\mathbf{X}}_{k-1|k-1}\middle|\boldsymbol{\theta}\right)\mathbf{V}_{k-1}\mathbf{A}\left(\hat{\mathbf{X}}_{k-1|k-1}\middle|\boldsymbol{\theta}\right)^T+\mathbf{Q}_k.\end{cases} \tag{10}$$

Filtering

Let $\hat{\mathbf{X}}_{k|k}$, $\mathbf{V}_k$ denote the conditional mean and conditional covariance of $\mathbf{X}_k$ given $\mathbf{Y}^k = \{\mathbf{Y}_1, ..., \mathbf{Y}_k\}$, then the filtered state is given as

$$\begin{cases} \hat{\mathbf{X}}_{k|k} = E\left\{\mathbf{X}_k|\mathbf{Y}^k\right\} = \hat{\mathbf{X}}_{k|k-1} + \mathbf{K}_k\hat{\boldsymbol{\Gamma}}_k \\ \mathbf{V}_k \quad = E\left\{\left(\mathbf{X}_k - \hat{\mathbf{X}}_{k|k}\right)\left(\mathbf{X}_k - \hat{\mathbf{X}}_{k|k}\right)^T\right\} \\ \qquad = \left[\mathbf{I} - \mathbf{K}_k\mathbf{C}\left(\hat{\mathbf{X}}_{k-1|k-1}|\boldsymbol{\theta}\right)\right]\mathbf{S}_k \\ \hat{\boldsymbol{\Psi}} \quad = E\left\{\hat{\boldsymbol{\Gamma}}_k\hat{\boldsymbol{\Gamma}}_k^T\right\} \\ \qquad = \mathbf{C}\left(\hat{\mathbf{X}}_{k-1|k-1}|\boldsymbol{\theta}\right)\mathbf{S}_k\mathbf{C}\left(\hat{\mathbf{X}}_{k-1|k-1}|\boldsymbol{\theta}\right)^T + \mathbf{R}_k \\ \mathbf{K}_k \quad = \mathbf{S}_k\mathbf{C}\left(\hat{\mathbf{X}}_{k-1|k-1}|\boldsymbol{\theta}\right)^T\hat{\boldsymbol{\Psi}}_k^{-1}. \end{cases} \tag{11}$$

3.3 Parameter estimation by the maximum likelihood method

From the estimated innovation $\hat{\boldsymbol{\Gamma}}_k$ and its covariance $\hat{\boldsymbol{\Psi}}_k$ with respect to the given parameters $\boldsymbol{\theta}$, the parameters $\boldsymbol{\theta}^*$ may be obtained by minimizing the (-2) log-likelihood function (9) as follows

$$\boldsymbol{\theta}^* = \arg\min_{\theta}\sum_{k=1}^{N}\left\{\log\left|\hat{\boldsymbol{\Psi}}_k(\theta)\right| + \hat{\boldsymbol{\Gamma}}_k(\boldsymbol{\theta})^T\hat{\boldsymbol{\Psi}}_k(\theta)^{-1}\hat{\boldsymbol{\Gamma}}_k(\theta)\right\} + 2N\log 2\pi$$

In this paper, the initial conditions $\mathbf{X}_{0|0}$ and $\mathbf{V}_0$ in the recursive Kalman filtering formulas (10–11), and the system observation noise variance $\mathbf{R}_k$ in DTMS model (7) are also regarded as the parameters to be estimated, which are all included in the parameter vector $\boldsymbol{\theta}$. The function 'FMINSEARCH' based on the Nelder-Mead method in the MATLAB Optimization Toolbox is used to carry out the parameter optimization in this paper.

4 Asset dynamic allocation control

If the DTMS model (7) is used to describe the dynamics of a currency exchange rate or a stock index in financial markets where P_k in model (7) denotes the exchange rate between two currencies or the price of a stock index, based on the estimated excess demand $\hat{\phi}_{k|k}$ rather than on the prediction of price, *i.e.* $\hat{P}_{k+1|k}$, dynamic asset allocation control may be carried out much more efficiently. Since the predictive error of $\hat{P}_{k+1|k}$ is usually close to a white noise due to the randomness of financial markets, it may be difficult to perform dynamic allocation satisfactorily based on predicted price. However, the information about future market trend offered by the directly-immeasurable excess demand ϕ_k possesses much better stability than the market trend information obtained from the prediction of price.

Without loss of generality, the JPY/USD (Japanese Yen/US Dollar) currency asset allocation control will be considered here, where P_k in model (7) denotes the JPY/USD exchange rate. The excess demand ϕ_k represents the market state, that is, if $\phi_k > 0$, it means the market is presently over-valued (or demand is now larger than supply, and '+' excess demand has pushed the price up), and one should sell yen and buy dollars; whereas, if $\phi_k < 0$, the market is presently under-valued (or demand is now smaller than supply, and '–' excess demand has pushed the price down), so one should buy yen and sell dollars in order to maintain or increase one's assets. From model (7), we may estimate the excess demand of the JPY/USD exchange market ϕ_k by $\hat{\phi}_{k|k}$ and use this price trend information to propose a dynamic asset allocation trading strategy for JPY/USD currency assets as follows

Assets allocation strategy for JPY/USD

$$\begin{cases} \text{if } \hat{\phi}_{k|k} > \tau_1, \text{ keep the Dollar 100\%;} \\ \text{if } \tau_2 < \hat{\phi}_{k|k} \le \tau_1, \text{ keep the Dollar 80\%, Yen 20\%;} \\ \text{if } -\tau_3 < \hat{\phi}_{k|k} \le \tau_2, \text{ keep the Dollar 50\%, Yen 50\%;} \\ \text{if } -\tau_4 < \hat{\phi}_{k|k} \le -\tau_3, \text{ keep the Dollar 20\%, Yen 80\%;} \\ \text{if } \hat{\phi}_{k|k} \le -\tau_4, \text{ keep the Yen 100\%.} \end{cases} \tag{12}$$

where $\tau = [\tau_1 \quad \tau_2 \quad \tau_3 \quad \tau_4]^T$ are the "optimum" switching threshold parameters.

Optimization of the threshold parameters

Define an asset-valuation function by

$$J(\tau) = -A_N(\boldsymbol{\tau}) + \frac{\mu}{N}\sum_{k=1}^{N}\left|A_k(\boldsymbol{\tau}) - \left[A_0 + \frac{k}{N}(A_N(\boldsymbol{\tau}) - A_0)\right]\right|^2 \tag{13}$$

where μ is a weighting factor, A_k represents assets at time k, and A_0 is the given initial assets. The first part of function (13) is the requirement for final assets, and the second part of function (13) represents the fluctuation amplitude of assets during the allocation control process. The optimal threshold parameters may be obtained by minimizing the valuation-function (13) as follows

$$\boldsymbol{\tau}^* = \arg\min_{\tau} J(\boldsymbol{\tau}) \tag{14}$$

which may be solved using the parameter optimization method mentioned earlier in Section 3.

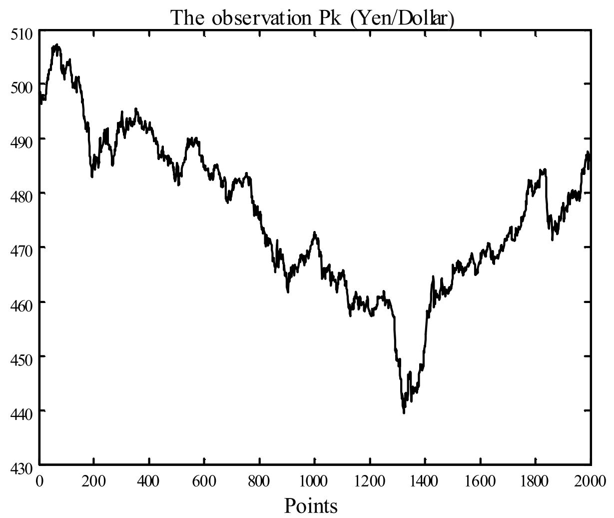

Fig. 1. The observation data (JPY/USD exchange-rate).

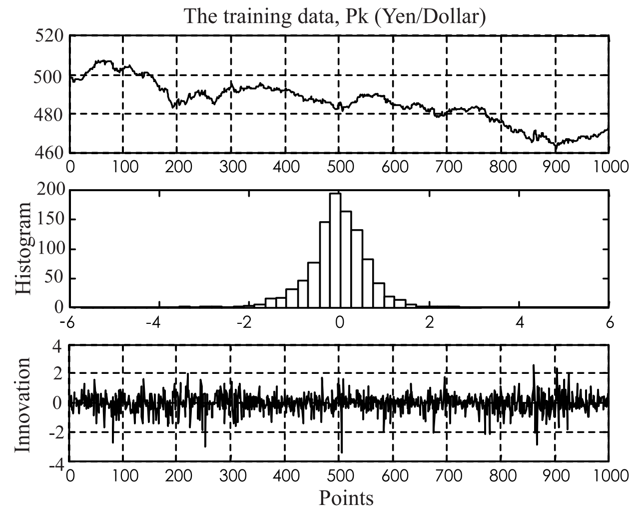

Fig. 2. P_k, estimated innovations and histogram of innovations for the training data (JPY/USD).

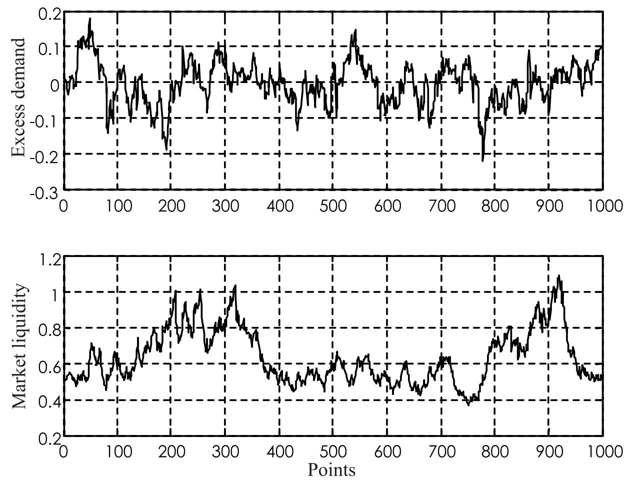

Fig. 3. The estimated excess demand $\hat{\phi}_{k|k}$ and liquidity $\hat{\lambda}_{k|k}$ for the training data (JPY/USD).

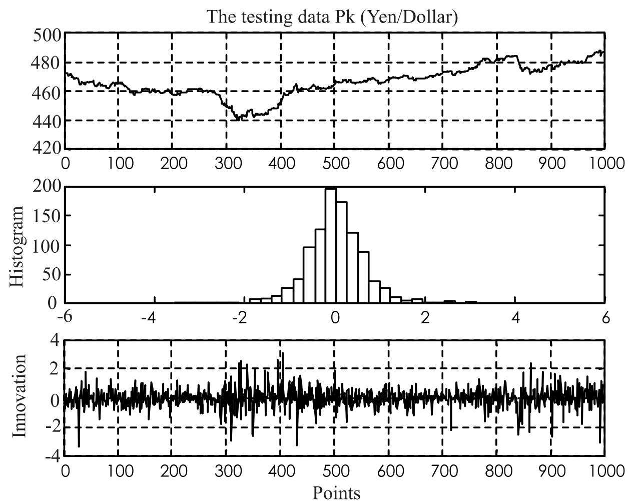

Fig. 4. P_k, estimated innovations and histogram of innovations for the testing data (JPY/USD).

5 Numerical study and financial implications

5.1 JPY/USD exchange rate time series modeling and JPY/USD currency asset allocation

The daily time series of the JPY/USD exchange rate from 17/01/1990 till 26/12/1997 is used to estimate the DTMS model (7) and to show the JPY/USD currency asset allocation control process. The observation data P_k is computed as follows

$$P_k = 100 \times \log Z_k \tag{15}$$

where Z_k denotes the (closing) spot price. Figure 1 shows the observations, in which the first 1 000 data points (training data) are used to estimate model (7) and to optimize the threshold parameters $\boldsymbol{\tau}$ for the asset allocation strategy (12), and the last 1 000 data points (testing data) are used to test the model and control performance. To avoid the problem of zero logarithms in the output variables $\mathbf{Y}_k = [P_k \quad \log(P_k - P_{k-1})^2]^T$ in model (7), approximation below suggested by Fuller [15] is used to cope with the observations

$$\log(P_k - P_{k-1})^2 \approx \log\left[(P_k - P_{k-1})^2 + \eta\sigma_P^2\right] - \frac{\eta\sigma_P^2}{(P_k - P_{k-1})^2 + \eta\sigma_P^2} \tag{16}$$

where σ_P^2 is the sampling variance of $P_k - P_{k-1}$, and η is a constant, which in this case study is set to 0.2.

The estimated results for model (7) using the method presented in Section 3 for the JPY/USD exchange rate training data and testing data are given in Figures 2–5 respectively. Figures 2 and 4 show that the estimated residuals of observations P_k both for training data and for testing data are very close to white noise, which shows that the estimates are valid. From Figures 3 and 5 we can see that the estimated excess demand $\hat{\phi}_{k|k}$ varies around zero, and is much smoother compared with the (strongly random)

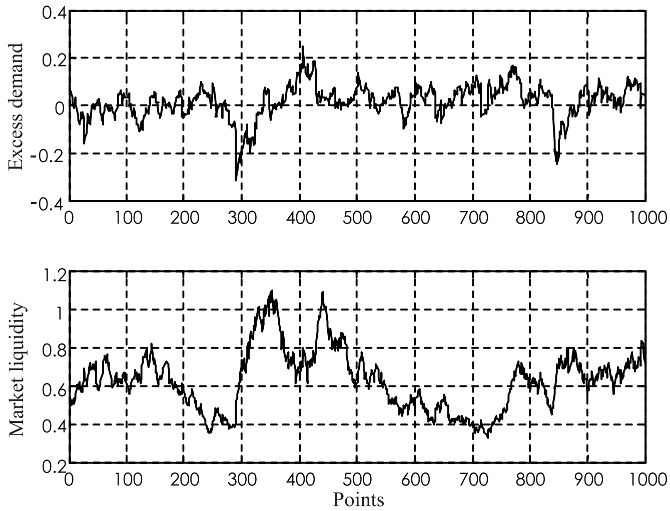

Fig. 5. The estimated excess demand $\hat{\phi}_{k|k}$ and the liquidity $\hat{\lambda}_{k|k}$ for the testing data (JPY/USD).

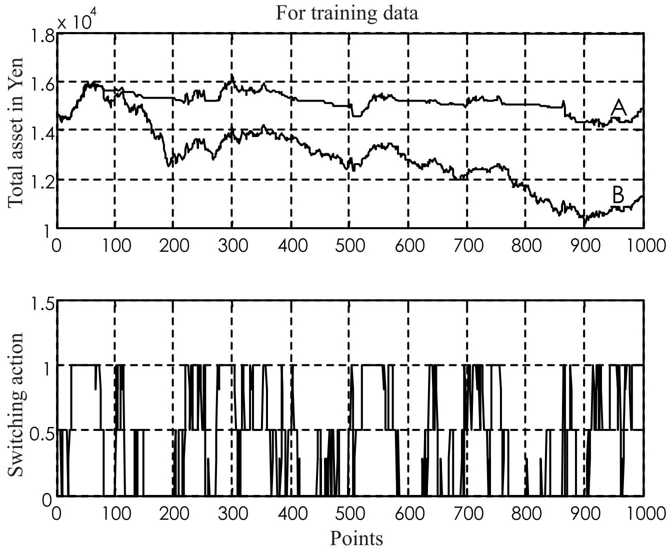

Fig. 6. Allocation control of currency assets computed in JPY for the training data (JPY/USD); the initial total assets are all \$100 (14,549 yen); the assets (A) with control have final assets \$132.01 (14,930 yen); the assets (B) without control have final assets \$100 (11,310 yen).

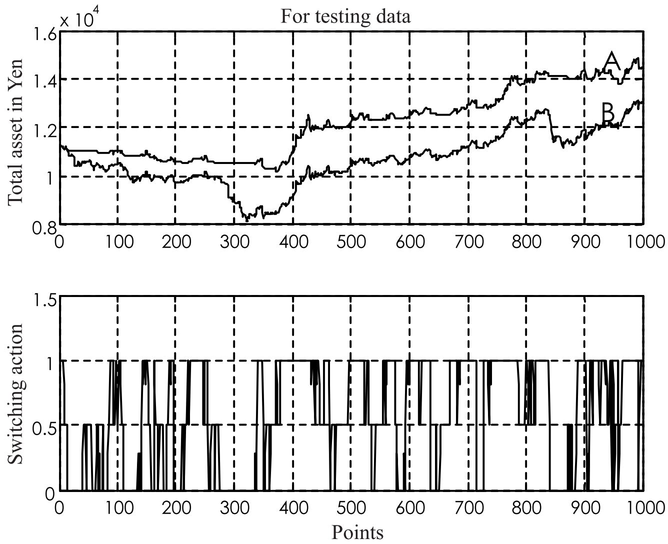

Fig. 7. Allocation control of currency assets computed in JPY for the testing data (JPY/USD); the initial total assets are all \$100 (11,275 yen); the assets (A) with control have final value \$111.052 (14,442 yen); the assets (B) without control have final value \$100 (13,005 yen).

estimated residuals of P_k shown in Figures 2 and 4. The estimated parameters and initial conditions of model (7) for the JPY/USD exchange rate are given as follows

$$\alpha_1 = 0.0008267, \quad 1+\beta_1 = 0.9417, \quad \gamma_1 = 0.05442,$$

$$\alpha_2 = 0.005910, \quad 1+\beta_2 = 0.9917, \quad \gamma_2 = 0.09417,$$

$$\gamma_3 = 1.001, \quad \delta = -2.253,$$

$$\varepsilon_1 = 0.0003169, \quad \varepsilon_2 = 2.513,$$

$$\mathbf{X}_{0|0} = \begin{pmatrix} 498.01 \\ -0.000166 \\ -1.4312 \end{pmatrix},$$

$$\mathbf{V}_0 = \begin{pmatrix} 0.879 & 0 & 0 \\ 0 & 1.035 & 0 \\ 0 & 0 & 0.875 \end{pmatrix} \times 10^{-9}.$$

Figures 6 and 7 show the JPY/USD currency asset allocation control results using strategy (12) for the training data and testing data respectively. In valuation function (13), the total assets to be controlled is computed in yen, which include the yen-assets and the dollar-asset equivalent to yen at any time, and the initial total assets are all set to be 100 dollars in each case. The switching actions plotted in Figures 6 and 7 show how the assets are switched, where '1' means keeping 100% dollars, '0' means keeping 100% yen, '0.82' means keeping 80% dollars and 20% yen, '0.28' means keeping 20% dollars and 80% yen, finally '0.5' means keeping 50% dollars and 50% yen. In Figures 6 and 7, the two B curves show the variation of the equivalent yen-assets corresponding to the fixed (\$100) dollar-assets varying with exchange rate in the case where there is no allocation control. We can see that by carrying out allocation control strategy (12), the total assets (A) effectively avoid the downward plunge of the dollar and prevent asset loss as shown in Figures 6 and 7. The optimum switching threshold parameters obtained by minimizing valuation function (13) using the training data (JPY/USD exchange rate), and other related parameters are given by

$$\tau_1 = 0.03212, \quad \tau_2 = 0.004378, \quad \tau_3 = 0.0007057,$$

$$\tau_4 = 0.01356, \quad N = 1000, \quad \mu = 1, \quad A_0 = \$100.$$

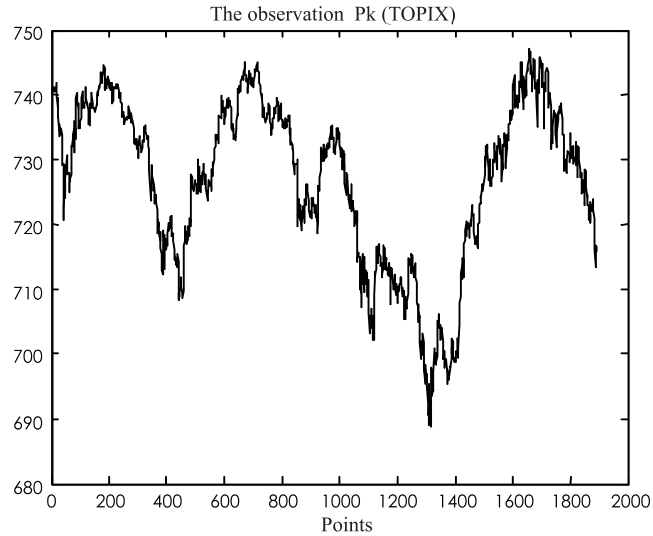

Fig. 8. The observation data (Japan TOPIX).

5.2 TOPIX (TOkyo stock Price IndeX) time series modeling and the stock/currency assets allocation

In this case study, we first identify a DTMS model (7) for the Japan TOPIX index time series and then carry out the stock/currency dynamic asset allocation according to the estimated excess demand of TOPIX which we consider as a special stock whose value per share is assumed to be just the TOPIX index value. Formula (15) is used here also for data conversion. The spot data Z_k in (15) is the daily closing price of the TOPIX index value time series from 1/10/1993 to 29/10/2000. The converted observation data P_k is shown in Figure 8, in which the first 1 000 data points are used as training data to estimate model (7) and to optimize the threshold parameters in trading strategy (12), and the last 891 data points are used as test data to check the modeling and allocation performance.

The modeling results using the DTMS model (7) for the training data and test data are given in Figures 9–12 respectively. Figures 9 and 11 show the modeling validity in terms of the estimated residuals of observation P_k. From Figures 9–12, it is clear that the estimated excess demand displays much more 'smoothness' than the estimated innovations as was also seen in Section 5.1 for the JPY/USD exchange rate time series modeling. This implies that the estimated excess demand information is more reliable and useful for making trading decisions than the estimated innovation. On the other hand, comparing Figure 3 (or Fig. 5) and Figure 10 (or Fig. 12), we can see that the fluctuations of market liquidity for the TOPIX are much faster than those of the JPY/USD exchange market. This means that the TOPIX stock market's index variation is larger than the JPY/USD exchange market's rate variation, which coincides with the actual situation. For example, the variance of daily return from 1/10/1993 to 29/10/2000 is 1.1778 for TOPIX index and 0.7809 for JPY/USD exchange rate.

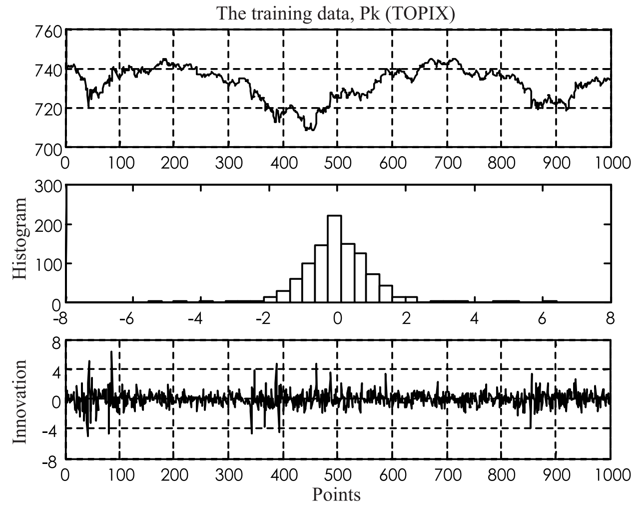

Fig. 9. P_k, estimated innovations and histogram of innovations for the training data (TOPIX).

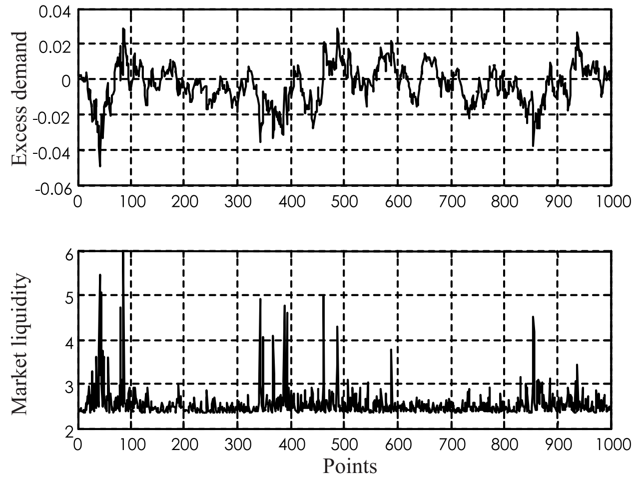

Fig. 10. The estimated excess demand $\hat{\phi}_{k|k}$ and the liquidity $\hat{\lambda}_{k|k}$ for the training data (TOPIX).

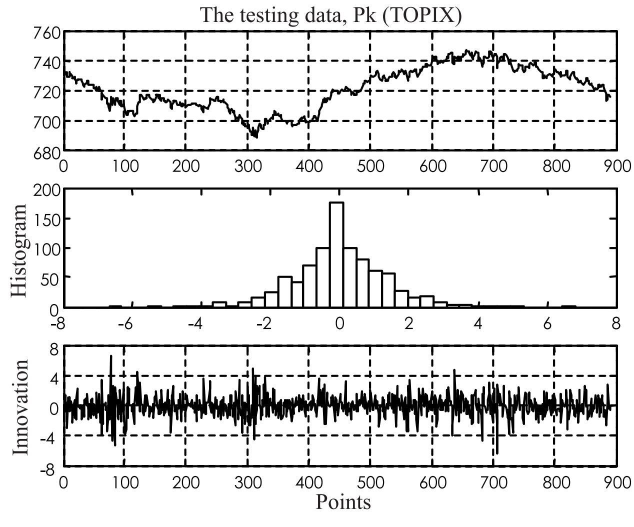

Fig. 11. P_k, estimated innovations and histogram of innovations for the testing data (TOPIX).

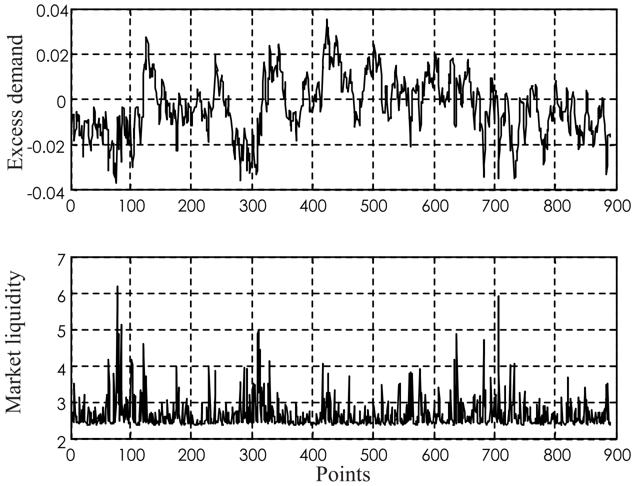

Fig. 12. The estimated excess demand $\hat{\phi}_{k|k}$ and the liquidity $\hat{\lambda}_{k|k}$ for the testing data (TOPIX).

Looking at these two examples, we can see that the discrete time microstructure (DTMS) model (7) proposed in this paper may be applied to describe different kind of financial markets. The estimated parameters and initial conditions of model (7) for the TOPIX are shown below

$$\alpha_1 = -0.0001837, \quad 1+\beta_1 = 0.9358, \quad \gamma_1 = 0.01672,$$
$$\alpha_2 = -0.004015, \quad 1+\beta_2 = 0.9828, \quad \gamma_2 = 0.3301,$$
$$\gamma_3 = 0.4381, \quad \delta = 3.117\times 10^{-4},$$
$$\varepsilon_1 = 4.025\times 10^{-4}, \quad \varepsilon_2 = 0.1385,$$

$$\mathbf{X}_{0|0} = \begin{pmatrix} 740.85 \\ 0.0020 \\ 0.5275 \end{pmatrix},$$

$$\mathbf{V}_0 = \begin{pmatrix} 0.5840 & 0 & 0 \\ 0 & 0.4286 & 0 \\ 0 & 0 & 3.676 \end{pmatrix} \times 10^{-5}.$$

We now consider the dynamic allocation problem where we have to choose to keep how many assets in TOPIX-based stock or put how many assets in currency according to the estimated excess demand of the TOPIX index. Assume here that the TOPIX-based stock has net asset value per share in JPY identical with the TOPIX index value, and the net asset value of TOPIX-based stock in JPY is the TOPIX index value multiplied by the number of shares. Figures 13 and 14 show the dynamic allocation control results where the allocation strategy (17) below used to allocate the dynamic assets composed of stock and currency is similar to strategy (12), and the meaning of the switching action given in Figures 13–14 is also similar to that used in Section 5.1 for the JPY/USD currency assets allocation.

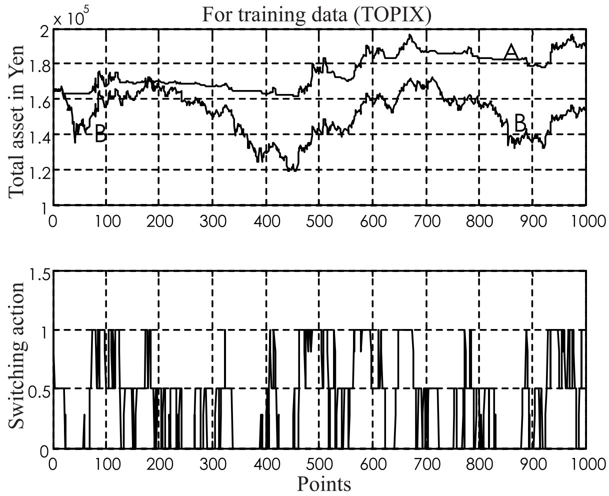

Fig. 13. Allocation control of stock/currency assets computed in JPY for the training data (TOPIX); the initial total assets are all 100 shares (163,409 yen); assets (A) with control have final value 123.424 shares (190,571 yen); assets (B) without control have final value 100 shares (154,404 yen).

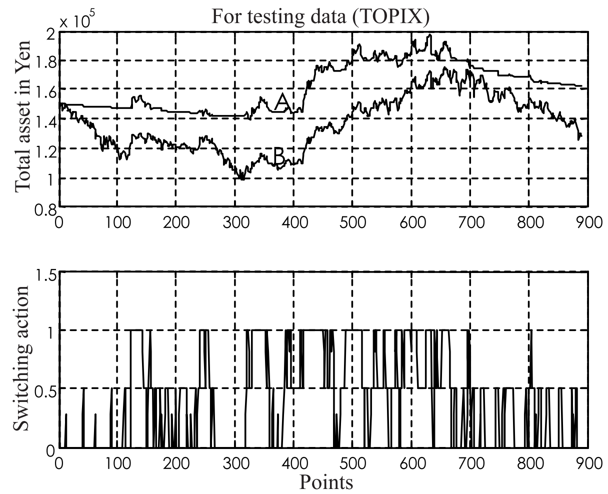

Fig. 14. Allocation control of stock/currency assets computed in JPY for the testing data (TOPIX); the initial total assets are all 100 shares (151,279 yen); assets (A) with control have final value 126.174 shares (161,966 yen); assets (B) without control have final value 100 shares (128,367 yen).

Assets allocation strategy for TOPIX

$$\begin{cases} \text{if } \hat{\phi}_{k|k} > \tau_1, \text{ keep the stock 100\%;} \\ \text{if } \tau_2 < \hat{\phi}_{k|k} \le \tau_1, \text{ keep the stock 80\%, currency 20\%;} \\ \text{if } -\tau_3 < \hat{\phi}_{k|k} \le \tau_2, \text{ keep the stock 50\%, currency 50\%;} \\ \text{if } -\tau_4 < \hat{\phi}_{k|k} \le -\tau_3, \text{ keep the stock 20\%, currency 80\%;} \\ \text{if } \hat{\phi}_{k|k} \le -\tau_4, \text{ keep the currency 100\%.} \end{cases} \tag{17}$$

From Figures 13–14 we can see results similar to the JPY/USD currency asset allocation control example shown in Section 5.1. Figures 13–14 show that the assets (A) controlled by the asset allocation procedure achieve an obvious increase compared with the assets (B) that are not controlled. The optimum switching threshold parameters in allocation strategy (17) obtained by minimizing valuating function (13) using the training data (TOPIX) and other related parameters are given as follows

$$\tau_1 = 0.005929, \quad \tau_2 = 0.0001188, \quad \tau_3 = 0.0001477,$$
$$\tau_4 = 0.005848, \quad N = 1000, \quad \mu = 1, \quad A_0 = 100.$$

6 Conclusions

The proposed discrete time microstructure model may effectively describe a class of stochastic volatility financial markets. The estimation method presented for the model on the basis of the Kalman filter and the likelihood maximum method may be easily used to estimate the model. The model used two important directly-immeasurable variables that represent the excess demand and the liquidity of financial market. Based on the estimated excess demand from the model, the proposed asset allocation control strategy provided an effective approach to increasing assets or preventing the loss of assets in rapidly varying financial markets. Further extensions to this work would be to improve the structure of the discrete time microstructure model, and to add a jump diffusion process for giving the model better adaptability to more general problems.

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